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Entropy of Additive Cellular Automata

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1. INTRODUCTION

One of the questions addressed in research on cellular automata is that of determining entropic properties for any given automaton. As with many other questions in recent cellular automata studies, this was initiated by Wolfram (1983), who provided formulas for certain automata defined over $\mathbf{Z}_2$. Also, Martin *et al.* (1984), Willson (1987*a,b*), and others have treated cellular automaton evolutions from a statistical mechanical point of view, and have computed similar formulas for the entropy of certain examplary automata, showing that the entropy decreases over time to a minimum. Discussions of entropy have also occurred in Vollmar's (1982) treatment of the efficient of polyautomata, and in Grassberger's (1986) paper on complexity measures for cellular automata. One interesting point which has arisen is a result of the demonstration by Lind (1984) that additive automata defined over infinite lattices do not have the entropy-decreasing properties of finite automata.

The purpose of this paper is to derive entropy formulas for all nearest neighbor additive cellular automata defined over $\mathbf{Z}_p$, where p is prime, to indicate which finite automata do and which do not decrease entropy, and to explain the reason that infinite automata do not decrease entropy.

Section 2 presents the general theory of nearest neighbor additive automata, formalized in terms of linear operators on a state space. Section 3 determines entropic properties of the automata studied in Section 2 in terms of properties of their defining operators. Section 4 discusses embeddings of finite automata in infinite automata and shows why no infinite automata have the entropy-decreasing property.

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2. OPERATOR APPROACH TO CELLULAR AUTOMATA

Let p be prime. A one-dimensional finite cellular automaton over $\mathbf{Z}_p$ is represented as a set of n sites located on a circle. Values at each site are chosen from $\mathbf{Z}_p$ according to an evolution rule which depends only on current site values. If the value at site i at time $t+1$ depends only on values at sites $i-1$, i, and $i+1$ at time t, the automaton follows a nearest neighbor rule. The set of p^n possible automaton states is denoted E_n and the automaton evolution rule is naturally represented as an operator Q: $E_n \to E_n$ (Voorhees, 1988). The automaton is denoted (Q, E_n). If, for all μ μ' in E_n $Q(\mu+\mu') = Q(\mu) + Q(\mu')$, the automaton is additive.

If μ_i is the value at site i for state μ, it is called the ith component of μ. The general component representation for an additive nearest neighbor operator Q is

$$[Q(\mu)]_i = x\mu_{i-1} + y\mu_i + z\mu_{i+1} \tag{2.1}$$

where x, y, z take values in $\mathbf{Z}_p$ and all sums and products are reduced modulo p, while all site indices are reduced modulo n. Thus the set $\{(x, y, z) \mid x, y, z \text{ in } \mathbf{Z}_p\}$ classifies all nearest neighbor additive cellular automata over $\mathbf{Z}_p$. There are p^3 distinct operators defined by (2.1).

If I is the identity on E_n while σ^{-1} and σ are, respectively, right and left shift operators defined by

$$\begin{aligned} [\sigma(\mu)]_i &= \mu_{i+1} \\ [\sigma^{-1}(\mu)]_i &= \mu_{i-1} \end{aligned} \tag{2.2}$$

then (2.1) is equivalent to the operator equation

$$Q = x\sigma^{-1} + yI + z\sigma \tag{2.3}$$

so that

$$Q^k = \sum_{r,s=0}^{r+s\leq k} T_{rs}^{(k+1)} x^r y^{k-r-s} z^s \sigma^{s-r} \tag{2.4}$$

where the $T_{rs}^{(k+1)}$ are the trinomial coefficients reduced modulo p. In component form (2.4) becomes

$$[Q^k(\mu)]_i = \sum_{r,s=0}^{r+s\leq k} T_{rs}^{(k+1)} x^r y^{k-r-s} z^s \mu_{i-r+s} \tag{2.5}$$

Table I lists all nearest neighbor additive cellular automata for $p=2$. In Table II a functional classification of nearest neighbor additive cellular automata is given and notation is introduced.

For binomial automata the trinomial coefficients are replaced by the binomial coefficients $\Pi_j^{(k+1)}$ drawn from the mod(p) Pascal triangle. The

Table I. Additive Operators for $p=2$

(x, y, z)	Operator	Component form
$(0,0,0)$	0^*	$[0^*(\mu)]_i = 0$
$(1,0,0)$	σ^{-1}	$[\sigma^{-1}(\mu)]_i = \mu_{i-1}$
$(0,1,0)$	I	$[I(\mu)]_i = \mu_i$
$(0,0,1)$	σ	$[\sigma(\mu)]_i = \mu_{i+1}$
$(1,1,0)$	$D^- = I + \sigma^{-1}$	$[D^-(\mu)]_i = \mu_i + \mu_{i-1}$
$(0,1,1)$	$D = I + \sigma$	$[D(\mu)]_i = \mu_i + \mu_{i+1}$
$(1,0,1)$	$\delta = \sigma + \sigma^{-1}$	$[\delta(\mu)]_i = \mu_{i-1} + \mu_{i+1}$
$(1,1,1)$	$\Delta = \sigma + I + \sigma^{-1}$	$[\Delta(\mu)]_i = \mu_{i-1} + \mu_i + \mu_{i+1}$

operator Q also has a representation as an $n \times n$ matrix:

$$Q^* = \begin{pmatrix} y & z & 0 & 0 & 0 & \cdots & 0 & x \\ x & y & z & 0 & 0 & \cdots & 0 & 0 \\ 0 & x & y & z & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ z & 0 & 0 & 0 & 0 & \cdots & x & y \end{pmatrix} \tag{2.6}$$

This is a circulant matrix for which the following results are relevant:

Theorem 1 (Bellman, 1960). Let $A = \text{circ}(a_0, \ldots, a_{n-1})$ be any $n \times n$ circulant matrix. Then the eigenvectors of A are

$$e_s = \begin{bmatrix} 1 \\ \boldsymbol{\omega}^s \\ \boldsymbol{\omega}^{2s} \\ \vdots \\ \boldsymbol{\omega}^{(n-1)s} \end{bmatrix}, \qquad s = 0, 1, \ldots, n-1 \tag{2.7}$$

where $\boldsymbol{\omega} = e^{2\pi i/n}$.

Further, the eigenvalues of A are

$$\lambda_s = \sum_{k=0}^{n-1} a_k \boldsymbol{\omega}^{ks}, \qquad s = 0, 1, \ldots, n-1 \tag{2.8}$$

Table II. Functional Classification of Additive Automata

A. Elementary automata: $(x, 0, 0)$, $(0, y, 0)$, and $(0, 0, z)$
  1. Nonsymmetric: $z\sigma$, $x\sigma^{-1}$
  2. Symmetric: yI

B. Binomially determined automata: $(x, y, 0)$, $(0, y, z)$, and $(x, 0, z)$
  1. Nonsymmetric: $D_{(x,y)} = yI + x\sigma$, $D^-_{(y,z)} = yI + z\sigma^{-1}$
  2. Symmetric: $\delta_{(x,z)} = x\sigma + z\sigma^{-1}$

C. Trinomial automata: (x, y, z)
  1. Symmetric: $\Delta_{(x,y,z)} = x\sigma + yI + z\sigma^{-1}$

Substitution in (2.8) shows that the eigenvalues of Q^* are given by the expression $\lambda_s = y + z\boldsymbol{\omega}^s + x\boldsymbol{\omega}^{(n-1)}s = x\boldsymbol{\omega}^{-s} + y + z\boldsymbol{\omega}^s$. This provides a direct connection with the algebraic approach of Martin *et al.* (1984). In that paper each state μ of E_n is represented by the polynomial

$$\mu(t) = \sum_{i=1}^{n} \mu_i t^{i-1} \tag{2.9}$$

while operators are represented as dipolynomials $Q(t) = xt^{-1} + y + zt$. In this representation automata evolution is generated by multiplication in a ring of polynomials: $\mu(t+1) = Q(t)\mu(t)$ with coefficients reduced modulo p and powers of t resolved modulo n. If the real variable t is replaced by $\boldsymbol{\omega}$, (2.9) becomes a cyclotomic polynomial in which the mod(n) requirement is automatically satisfied since $\boldsymbol{\omega}^n = 1$. The cyclotomic polynomial $Q(\boldsymbol{\omega})$ is, by (2.8), just λ_1—the second eigenvalue of the matrix Q^*. Since the mapping $Q^* \leftrightarrow Q(\boldsymbol{\omega})$ is a ring isomorphism (Davis, 1979), the formalism of Martin *et al.* is a particular representation, in terms of real polynomial rings, of the more general operator approach presented in this paper. The advantages of the more general formalism are that it connects directly with forms of thought found in classical and quantum physics, and that it objectifies automata transition rules as abstract objects which can be formally manipulated across a varity of different representations (e.g., in component form, as circulant matrices, as cyclotomic—or other—polynomials, etc.)

The notation $D_{(x,y)}^{-}$, $D_{(y,z)}$, $\delta_{(x,z)}$, and $\Delta_{(x,y,z)}$ introduced in Table II will be used to denote operators corresponding, respectively, to $(x, y, 0)$, $(0, y, z)$, $(x, 0, z)$, and (x, y, z).

An automaton (Q, E_n) can be geometrically represented by a state transition diagram. This is a directed graph in which each vertex represents an element of E_n and there is an edge directed from a vertex A to a vertex B if and only if the corresponding states μ_A and μ_B satisfy $Q(\mu_A) = \mu_B$. In this case μ_A is a predecessor of μ_B. A state may have more than one predecessor, but all states map to a single successor. That is, each vertex of the state transition diagram may have many lines directed into it, but has only a single line directed out.

3. PREDECESSOR STATES AND ENTROPY

In this section the numbers of predecessor states for all finite nearest neighbor additive cellular automata are computed and the entropies of these automata as a function of time (i.e., iteration step) are derived. This is a generalization of work carried out by Wolfram (1983) and Martin *et al.* (1984). The in degree of a vertex of the state transition diagram for an automaton (Q, E_n) is defined as the number of edges directed into that vertex. The out degree, the number of edges directed out of a vertex, is always 1. Martin *et al.* prove the following.

Lemma 1. Trees rooted at all vertices of all cycles of the state transition diagram of an additive cellular automaton are isomorphic to the tree rooted at $\mathbf{0}$.

Lemma 2. Two states of an additive automaton (Q, E_n) map to the same state after a single iteration if and only if they differ by a state which maps to $\mathbf{0}$.

An immediate consequence of these two lemmas is the following.

Lemma 3. Let (Q, E_n) be any additive automaton. The in degrees of all vertices with predecessors in the state transition diagram are equal, taking $Q(\mathbf{0}) = \mathbf{0}$ as contributing to the in degree of $\mathbf{0}$.

As a result of this, the in degree of states having predecessors is an invarient for an automaton, and can be computed by counting the number of solutions to the homogeneous equation $Q(\mu) = \mathbf{0}$. Since every state of an elementary automaton is on a cycle, the in degree for these automata is 1. For binomial and trinomial automata the case is more interesting.

For automata $(D_{(y,z)}, E_n)$ the equation $D_{(y,z)}(\mu) = \mathbf{0}$ is written

$$\begin{aligned} y\mu_1 + z\mu_2 &= 0 \\ y\mu_2 + z\mu_3 &= 0 \\ &\cdots \\ y\mu_{n-1} + z\mu_n &= 0 \\ y\mu_n + z\mu_1 &= 0 \end{aligned} \tag{3.1}$$

These equations have solution $\mu_i = (p - z^{-1}y)^{i-1}\mu_1$ with the constraint $(p - z^{-1}y)^n = 1 \bmod(p)$. If this constraint is satisfied, there are p possible solutions corresponding to $\mu_1 = 0, 1, 2, \ldots, p-1$. If it is not satisfied, the only solution is $\mu = \mathbf{0}$.

Noting the symmetry between $D_{(y,z)}$ and $D^-_{(x,y)}$, we have expressions for the in degree of both of these automata:

Theorem 2. The in degree of states of $(D_{(y,z)}, E_n)$ and $(D^-_{(x,y)}, E_n)$ is 1 unless $(p - z^{-1}y)^n \equiv 1 \bmod(p)$ [respectively, $(p - x^{-1}y)^n \equiv 1 \bmod(p)$], in which case it is p.

Corollary 2a. If $p = 2$, the in degrees of (D, E_n) and (D^-, E_n) are always 2.

Corollary 2b. The in degrees of $(D_{(y,p-y)}, E_n)$ and $(D^-_{(p-y,y)}, E_n)$ are always p.

Proof. We consider $(D_{(y,p-y)}, E_n)$. For this case $z = p - y$. Let $p - (p-y)^{-1}y = K$. Then $p(p-y) - y = K(p-y)$, which implies that $-y = -Ky$, or $K = 1 \bmod(p)$. ■

Writing out $\delta^{(\mu)}_{(x,z)} = \mathbf{0}$ gives the set of equations

$$\begin{aligned} x\mu_n + z\mu_2 &= 0 \\ x\mu_1 + z\mu_3 &= 0 \\ &\cdots \\ x\mu_{n-2} + z\mu_n &= 0 \\ x\mu_{n-1} + z\mu_1 &= 0 \end{aligned} \tag{3.2}$$

If n is even, the solutions of these equations are

$$\begin{aligned} \mu_{2i} &= (p - z^{-1}x)^{i-1}\mu_2, \qquad i = 1, \ldots, n/2 \\ \mu_{2i-1} &= (p - z^{-1}x)^{i-1}\mu_1, \qquad i = 1, \ldots, n/2 \end{aligned} \tag{3.3}$$

with the constraint $(p - z^{-1}x)^n = 1 \bmod(p)$. If this constraint is satisfied, there are p^2 solutions and if it is not satisfied, $\mathbf{0}$ is the only solution. If n is odd, the solution of (3.2) is given in terms of μ_1 only:

$$\begin{aligned} \mu_{2i-1} &= (p - z^{-1}x)^{i-1}\mu_1 \\ \mu_{2i} &= (p - z^{-1}x)^{i+(n-1)/2}\mu_1 \end{aligned} \tag{3.4}$$

subject to the same constraint as in the n-even case. This is summarized in the following result.

Theorem 3. If n is even, the automaton $(\delta_{(x,z)}, E_n)$ will have in degree p^2 or 1, and if n is odd, the in degree will be p or 1, in both cases depending on whether or not $(p - z^{-1}x)^n \equiv 1 \bmod(p)$.

Corollary 3a (Theorem 3.2 of Martin, *et al.*). For $p = 2$ the in degree of $(\delta_{(x,z)}, E_n)$ is four if n is even and two if n is odd.

Corollary 3b. The in degree of $(\delta_{(x,p-x)}, E_n)$ is p^2 if n is even and p if n is odd.

For the trinomial automata the equations $\Delta_{(x,y,z)}(\mu) = \mathbf{0}$ become

$$\begin{aligned} x\mu_n + y\mu_1 + z\mu_2 &= 0 \\ x\mu_1 + y\mu_2 + z\mu_3 &= 0 \\ &\cdots \\ x\mu_{n-2} + y\mu_{n-1} + z\mu_n &= 0 \\ x\mu_{n-1} + y\mu_n + z\mu_1 &= 0 \end{aligned} \tag{3.5}$$

These have solutions in terms of μ_1 and μ_2, assuming satisfaction of constraints:

Theorem 4. The solution of (3.5) is given by $\mu_i = A_i\mu_1 + B_i\mu_2$ with $A_1 = B_2 = 1$, $A_2 = B_1 = 0$, while for $i \geq 3$ and $\{a\}$ the least integer greater than or equal to a,

$$A_i = \sum_{j=0}^{\{(i-3)/2\}} \Pi_{i-2j-2}^{(i-j-2)}(p - z^{-1}x)^{j+1}(p - z^{-1}y)^{i-2j-3}$$
$$B_i = \sum_{j=0}^{\{(i-2)/2\}} \Pi_{i-2j-1}^{(i-j-1)}(p - z^{-1}x)^{j}(p - z^{-1}y)^{i-2j-2} \tag{3.6}$$

subject to constraints $A_{n+1} = 1$, $B_{n+1} = 0$. If these constraints are satisfied, there are p^2 solutions. If they are not satisfied, the only solution is $\mathbf{0}$.

Proof. Equations (3.5) yield $\mu_i = (p - z^{-1}x)\mu_{i-2} + (p - z^{-1}y)\mu_{i-1}$. This is iterated backward until μ_i is expressed in terms of μ_1 and μ_2. The coefficients A_i and B_i are found to satisfy recursion relations

$$A_i = (p - z^{-1}x)B_{i-1}, \qquad B_i = (p - z^{-1}y)B_{i-1} + A_{i-1}$$

and the constraints arise through the periodic boundary conditions. ∎

Corollary 4. If $p = 2$, then (Δ, E_n) has in degree 1 unless $3 | n$, in which case the in degree is 4.

Since the entropy of a cellular automaton gives a measure of the number of reachable states at any point in the automaton evolution, information on in degrees can be used to study the entropy-decreasing property.

Lemma 4. Let (Q, E_n) be a finite nearest neighbor additive cellular automaton and denote the in degree of a state μ by $d^{(i)}(\mu)$, taking this as 0 for states not having predecessors. Then

$$\sum_{\mu \in E_n} d^{(i)}(\mu) = p^n \tag{3.7}$$

Proof. The out degree of every state is equal to 1, so that the sum over out degrees is the number of states in E_n, which is p^n. But every edge of the state transition diagram directed out of a vertex must terminate on some other vertex, and every edge directed into a vertex must have originated on a vertex. Hence the sum of in degrees must equal the sum of out degrees. ∎

Lemma 5. Let the matrix Q^* have nullity $\nu(Q^*)$. Then for all μ in E_n having predecessors, $d^{(i)}\mu = p^{\nu(Q^*)}$.

Proof. If μ has predecessors, then $d^{(i)}(\mu) = d^{(i)}(\mathbf{0})$ and this is just the number of solutions of $Q^*(\mu) = \mathbf{0}$. The number of components of μ which act as free parameters in the general solution is, by standard theorems of linear algebra, $\nu(Q^*)$. Each free parameter can take values $0, 1, \ldots, p-1$, so the lemma follows. ■

Let $E_n^{(0)}$ be the subset of states in E_n which do not have predecessors, $E_n^{(p)}$ be the subset of states having predecessors, with $N^{(0)}$ and $N^{(p)}$ the respective numbers of states in these two sets. By Lemma 4, $d^{(i)}N^{(p)} = p^n$. Hence, by Lemma 5,

$$\begin{aligned} N^{(p)} &= p^{n-\nu(Q^*)} \\ N^{(0)} &= p^n(1-p^{-\nu(Q^*)}) \end{aligned} \tag{3.8}$$

and the fractions of states in $E_n^{(0)}$ and $E_n^{(p)}$ are given by

$$\begin{aligned} \%(E_n^{(p)}) &= p^{-\nu(Q^*)} \\ \%(E_n^{(0)}) &= 1-p^{-\nu(Q^*)} \end{aligned} \tag{3.9}$$

Taking $a(n)$ as the maximum tree height, Lemmas 1 and 3 indicate that the p^n states of (Q, E_n) will partition into sets of $[d^{(i)}]^{a(n)}$ states, each set constituting a tree rooted on a cycle or fixed point of Q. Thus, we have the following result.

Theorem 5. The number of states on cycles of (Q, E_n), counting fixed points as cycles of period 1, is given by $p^{n-a(n)\nu(Q^*)}$. The number of states which can have no predecessors after t iterations is $p^{n-t\nu(Q^*)}$ for $t < a$.

On the basis of this theorem, the entropy formulas of Wolfram (1983) and Martin *et al.* (1984) can be generalized. Taking the normalized measure entropy

$$S(t) = (n \log_2 p)^{-1} \sum_{\mu \in E_n} P_t(\mu) \log_2 P_t(\mu) \tag{3.10}$$

where $P_t(\mu)$ is the probability of state μ after t iterations of Q, yields the entropy evolution formula

$$S(t) = \begin{cases} 1 - t\nu(Q^*)/n, & 0 \le t < a(n) \\ 1 - a(n)\nu(Q^*)/n, & t \ge a(n) \end{cases} \tag{3.11}$$

The fractional change in $S(t)$ per time step for $t < a(n)$ is given by $a^{-1}[S(t+1) - S(t)] = -\nu(Q^*)/na(n)$. Thus, the nullity of the matrix Q^* appears as a discrete analogue of the rate of change of entropy. Noting that Lemma 5 implies $\nu(Q^*) = \log_p d^{(i)}$, and recalling the results of Theorems 3–5, one obtains formulas for the nullity of the various nearest neighbor additive operators:

Theorem 6.

$$\nu(D_{(y,z)}) = \nu(D^-_{(x,y)}) = \begin{cases} p, & (p - z^{-1}y)^n \equiv 1 \bmod(p) \\ 0, & \text{otherwise} \end{cases} \tag{3.12}$$

$$\nu(\delta_{(x,z)}) = \begin{cases} p^2, & n \text{ even}, \quad (p - z^{-1}x)^n \equiv 1 \bmod(p) \\ p, & n \text{ odd}, \quad (p - z^{-1}x)^n \equiv 1 \bmod(p) \\ 0, & \text{otherwise} \end{cases} \tag{3.13}$$

$$\nu(\Delta_{(x,y,x)}) = \begin{cases} p^2, & A_{n+1} = 1, \quad B_{n+1} = 0 \\ 0, & \text{otherwise} \end{cases} \tag{3.14}$$

As a result of this theorem, only certain cellular automata are seen to decrease entropy, namely those for which $\nu(Q^*) \neq 0$. By (3.11), the entropy for these automata, with $n = p^m n_0$ and $a(n) = p^m$, evolves according to the formula

$$S(t) = \begin{cases} 1 - t\nu(Q^*)/p^m n_0, & 0 \leq t < p^m \\ 1 - \nu(Q^*)/n_0, & p^m \leq t \end{cases} \tag{3.15}$$

and the final value of this entropy is independent of m.

4. DISCUSSION

In the previous section only finite automata have been considered. Certain of these automata, namely those for which $\nu(Q^*) \neq 0$, were shown to have the property that their entropy decreased, reaching a lower bound after a finite number of iterations. Lind (1984), on the other hand, has demonstrated that all infinite additive cellular automata have constant entropy. This seems to present a contradiction, since a finite automaton (Q, E_n) can be naturally embedded in the infinite automaton (Q, E^+) by taking the subset E_n^+ of E^+ consisting of all periodic elements with period a divisor of n. Further, at least for those operators which are not dependent on σ^{-1}, the cycle properties of this embedded automaton are the same as for the original. That is, if the entropy of (Q, E_n) decreases with automaton evolution, then the entropy of (Q, E_n^+) will also decrease. Further, this property is independent of n. That is, the entropy of (Q, E^+) remains constant even though the entropy, evaluated over the set $\bigcup_{\text{all } n} E_n^+$ of all periodic elements of E^+, decreases.

The resolution of this question follows by noting that E^+ maps to the unit interval $[0, 1]$ when elements are taken as binary expansions in powers of 2^{-1}. With this mapping all periodic or eventually periodic sequences map to rational numbers, and all other sequences map to irrationals. It can be shown (Voorhees, 1989) that only those sequences which are eventually periodic can be predecessors of states on cycles. The entropy-decreasing property of finite automata is a manifestation of the fact that for certain of these automata not all states lie on cycles, but all states map to cycles. Since entropy is computed over the ensemble of all states, the entropy of an infinite automaton does not decrease, because the states corresponding to irrational numbers in [0, 1] do not map to cycles. That is, entropy decreases on at best a countable dense subset of E^+, and this, having measure zero, does not contribute to the total ensemble average. On the other hand, since all binary computations involve only terminating sequences, entropy-decreasing properties on this subset may be of significance.

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